Exercises Chapter V

Mathematical Methods of Bioengineering

April 26, 2021

Contents

This is a guide to the exercises that you can solve. If you fall short, there are more similar exercises in the books of the subject. You can ask me questions at the end of class or in tutoring.

The idea is that you solve the exercises and do them on the blackboard in class. Each time you go to the board, it will count to the 5% of the final mark. You must participate at least 3 times in order to get the full 5% and at least 6 times to raise the final grade by +0.5 points.

1 Vectors

2 Differentiation in Several Variables

3 Vector Valued Functions

4 Maxima and Minima in Several Variables

5 Multiple Integration

5.1 Introduction: Areas and Volumes

1. Evaluate the iterated integral given.

(a) $\int_0^2 \int_1^3 (x^2 + y)\, dy\, dx$

(b) $\int_0^{\pi/2} \int_0^1 e^x \cos y\, dx\, dy$

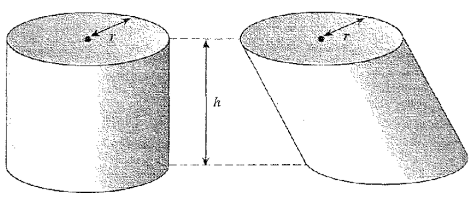
(c) $\int_1^9 \int_1^e \frac{\ln \sqrt{x}}{xy}\, dx\, dy$

2. Find the volume of the region that lies under the graph of the paraboloid $z = x^2 + y^2 + 2$ and over the rectangle $R = \{(x, y)| -1 \leq x \leq 2, 0 \leq y \leq 2\}$ in two ways:

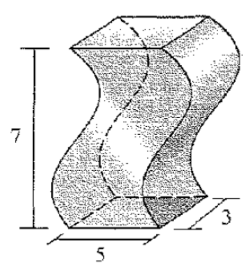
    (a) By using Cavalieri's principle to write the volume as an iterated integral that results from slicing the region by parallel planes of the form $x =$ constant.

    (b) By using Cavalieri's principle to write the volume as an iterated integral that results from slicing the region by parallel planes of the form y = constant.

3. Use the Cavalieri's principle to demonstrate that the volumes of two cylinders with equal base an height are equal (see figure).



4. Using the Cavalieri's principle, find the volume of the structure shown in the next figure where each transversal section is a rectangle with longitude 5 and with 3.



5.2 Double Integrals

1. Evaluate the given iterated integrals. In addition, sketch the regions D that are determined by the limits of integration.

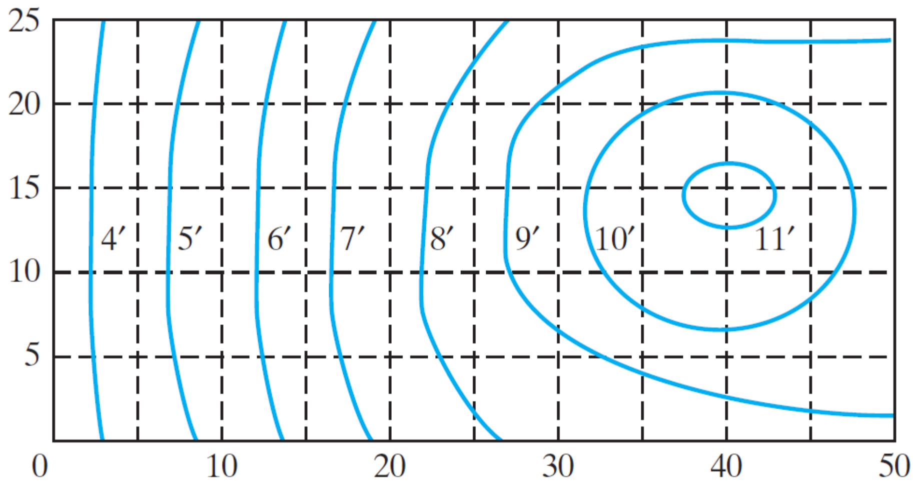
    (a) $\int_{-1}^3 \int_x^{2x+1} xy\, dy\, dx$

    (b) $\int_0^4 \int_0^{2\sqrt{y}} x \sin y^2\, dx\, dy$

    (c) $\int_0^\pi \int_0^{\sin x} y \cos x\, dy\, dx$

(d) $\int_0^1 \int_{-e^x}^{e^x} y^3 \, dy \, dx$

2. The next figure shows the level curves indicating the varying depth (in feet) of a 25 ft by 50 ft swimming pool. Use a Riemann sum to estimate the volume of water that the pool contains.



3. Integrate the function $f(x, y) = 1 - xy$ over the triangular region whose vertices are $(0, 0), (2, 0), (0, 2)$.

4. Integrate the function $f(x, y) = x + y$ over the region bounded by $x + y = 2$ and $y^2 - 2y - x = 0$.

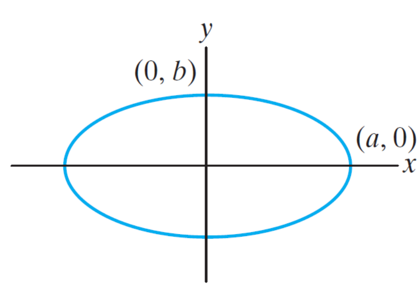
5. Evaluate $\int\int_D 3y \, dA$, where D is the region bounded by $xy^2 = 1$, $y = x$, $x = 0$, and $y = 3$.

6. Let D be an elementary region in $\mathbb{R}^2$.

   (a) Use the definition of the double integral to explain why $\int\int_D 1 \, dA$ gives the area of D.

   (b) Use part a) to show that the area inside a circle of radius a is πa^2.
   Hint: make the change of variable $x = a \sin t$ and use that $\cos^2(t) = \frac{1+\cos 2t}{2}$.

7. Use double integrals to calculate the area inside the ellipse whose semiaxes have lengths a and b.



8. Use double integrals to find the area of the region bounded by the parabola $y = 2 - x^2$, and the lines $x - y = 0$, $2x + y = 0$.

9. Find the volume under the plane $z = 4x + 2y + 25$ and over the region in the xy-plane bounded by $y = x^2 - 10$ and $y = 31 - (x - 1)^2$.

5.3 Changing the Order of Integration

1. Consider the integral

$$\int_0^2 \int_{x^2}^{2x} (2x+1)dy\ dx$$

   (a) Evualuate this integral.

   (b) Sketch the region of integration.

   (c) Write an equivalent iterated integral with the order of integration reversed. Evaluate this new integral and check that your answer agrees with part a).

2. Calculate:

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} dx\ dy$$

3. Calculate:

$$\int_0^3 \int_0^{9-x^2} \frac{xe^{3y}}{9-y} dy\ dx$$

4. Calculate:

$$\int_0^2 \int_{y/2}^1 e^{-x^2} dx\ dy$$

5.4 Triple Integrals

1. Evaluate the integrals:

   (a) $\int\int\int_{[0,1]\times[0,2]\times[0,3]} x^2 + y^2 + z^2 dV$

   (b) $\int\int\int_{[1,e]\times[1,e]\times[1,e]} \frac{1}{xyz}\ dV$

   (c) $\int_{-1}^2 \int_1^{z^2} \int_0^{y+z} 3yz^2\ dxdydz$

   (d) $\int_1^3 \int_0^z \int_1^{xz} (x+2y+z)\ dydxdz$

2. Consider a ball of radius r:

   (a) Write the volume of the ball of radius r as a triple integral. Do not calculate the triple integral.

   (b) Use the formula of the volume of a *solid of revolution*, $V = \pi \int_a^b f(x)^2\ dx$, to find that the volume of a sphere is $4\pi r^3/3$.

3. Find the volume of the solid bounded by $z = 4 - x^2$, $x + y = 2$, and the coordinate planes.

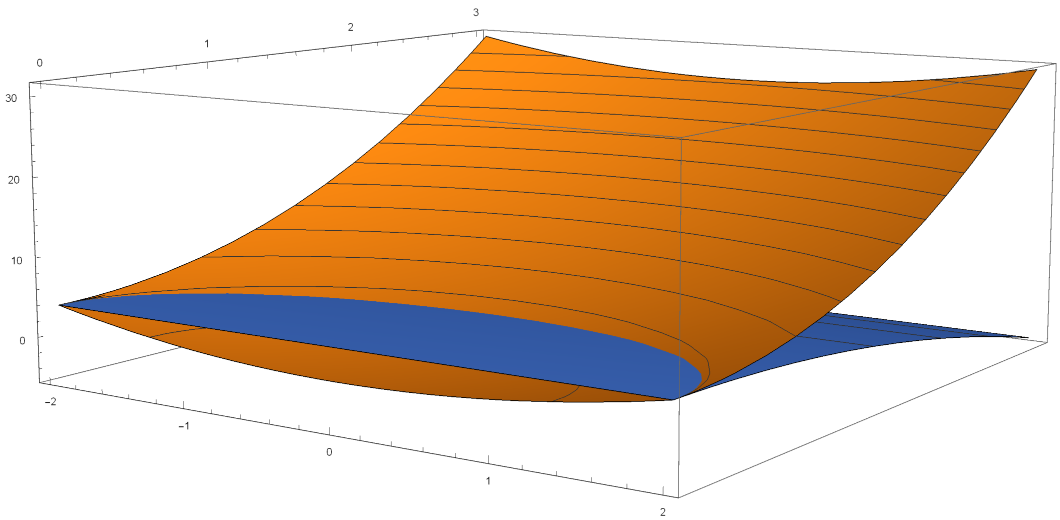


Figure 1: The functions $z = x^2 + 3y^2$ and $z = 4 - y^2$ from exercises 4.

4. Consider the iterated integral

$$\int_{-2}^{2} \int_{0}^{\frac{1}{2}\sqrt{4-x^2}} \int_{x^2+3y^2}^{4-y^2} (x^3 + y^3)\ dzdydx$$

   (a) This integral is equal to a triple integral over a solid region $\mathbf{W}$ in $\mathbb{R}^3$. Describe $\mathbf{W}$.

   (b) Set up an equivalent iterated integral by integrating first with respect to z, then with respect to x, then with respect to y. Do not evaluate your answer.

   (c) Set up an equivalent iterated integral by integrating first with respect to x, then with respect to z, then with respect to y. Do not evaluate your answer.

5. Suppose that the temperature at a point in the cube $W = [-1, 1] \times [-1, 1] \times [-1, 1]$ varies in proportion to the square of the point's distance from the origin.

   (a) What is the average temperature of the cube?

   (b) Describe the set of points in the cube where the temperature is equal to the average temperature.

   Note: The mean value of a function over a region W can be computed as: $f_{\text{avg}} = \frac{1}{V(W)} \int\int\int_W f\ dW$

5.5 Change of variable

1. Let $\mathbf{T}(u, v) = (3u, -v)$.

   (a) Write $\mathbf{T}(u, v) = A \cdot [u, v]^t$ for a suitable matrix A.

   (b) Describe the image $D = T(D^*)$ where D^* is the unit square $[0, 1] \times [0, 1]$.

2. OPTIONAL: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ by the transformation given by:

$$T(\rho, \varphi, \theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$$

(a) Determine $D = \mathbf{T}(D^*)$, where $D^* = [0,1] \times [0,\pi] \times [0,2\pi]$.

(b) Determine $D = \mathbf{T}(D^*)$, where $D^* = [0,1] \times [0,\pi/2] \times [0,\pi/2]$.

(c) Determine $D = \mathbf{T}(D^*)$, where $D^* = [\frac{1}{2},1] \times [0,\pi/2] \times [0,\pi/2]$.

3. This problem concerns the iterated integral

$$\int_0^1 \int_{y/2}^{(y/2)+2} 2x - y \; dxdy$$

(a) Evaluate this integral and sketch the region D of integration in the xy-plane.

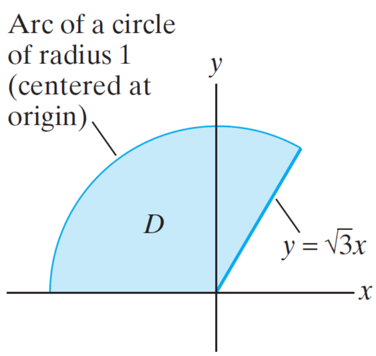
(b) Let $u = 2x - y$ and $v = y$. Find the region D^* in the uv-plane that corresponds to D.

(c) Use the change of variables theorem to evaluate the integral by using the substitution $u = 2x - y$, $v = y$.

4. Evaluate

$$\int\int_D \cos(x^2 + y^2) \; dA$$

where D is the shaded region in the figure below.



5. OPTIONAL: Evaluate

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} \frac{e^z}{\sqrt{x^2+y^2}} \; dzdydx$$

using cylindrical coordinates.

6. In this exercises we try to compute $\int_0^\infty e^{-x^2} dx$. The indefinite integral doesn't have an expression in term of elemental functions, nevertheless, using an appropriate change of variables is possible to compute the above integral.

(a) Consider the region D that is the portion of the disk of radius r that lies in the first quadrant. Compute

$$\int\int_D e^{-x^2-y^2} \, dx$$

Hint: change to polar coordinates system.

(b) Show that $\int_0^L \int_0^L e^{-x^2-y^2} dx\ dy = (\int_0^L e^{-x^2} dx)^2$.

(c) Compute $\displaystyle\int_0^{\infty} e^{-x^2} dx = \lim_{L\to\infty} \int_0^L e^{-x^2} dx$.

7. Using previous exercise check that $f_X(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is a probability density function, where $X \sim N(0,1)$. That's it to check that $f(x) \geq 0$ and $\int_{\mathbb{R}} f(x) dx = 1$.